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THE CARDIOID AND TRICUSPID: QUARTICS WITH THREE CUSPS

BY R. C. ARCHIBALD

1. THE *cardioid* and the three-cusped hypocycloid or *tricuspid* are of interest not only on account of the great many beautiful properties which they possess, but also because of their historic association with the names of many eminent mathematicians.

The cardioid was discussed as an epicycloid by Jacob Ozanam as early as 1691,[*] and later by the Bernoullis, de L' Hôpital, L. Carré (author of the first complete work on the integral calculus, 1700), de Réaumur, de la Hire, Maclaurin, Castilleoneus, Euler, Cramer, Quetelet (who showed the uses of the curve in graphic astronomy), Magnus (who showed the curve to be an inverse of the parabola), J. C. Maxwell, R. Proctor, Wolstenholme, Weill, Laguerre, Brocard, etc.

The tricuspid seems to have been first conceived, as a unicursal quartic, by Euler,[†] in 1745, in the treatment of a problem in catacaustics. Some idea of the interest which this curve has excited in the minds of many of the most noted modern mathematicians, and of the enormous number of papers devoted to the discussion of it, may be obtained by glancing through the bibliographical lists in *L'Intermédiaire des Mathématiciens*, vol. 3 (1896) and vol. 4 (1897).

The object of the present paper is to show the intimate relation which exists between these apparently entirely dissimilar curves, and, in illustration of the theory of projection, to make this relation a basis for the derivation of some theorems for the general tricuspidal quartic. Incidentally, we shall see how a comparatively simple property of one curve may be made use of to prove a property of the other which could be otherwise derived only by much longer and less elegant methods.

The subject has been touched upon[‡] by Siebeck, Frahm, Wolstenholme,

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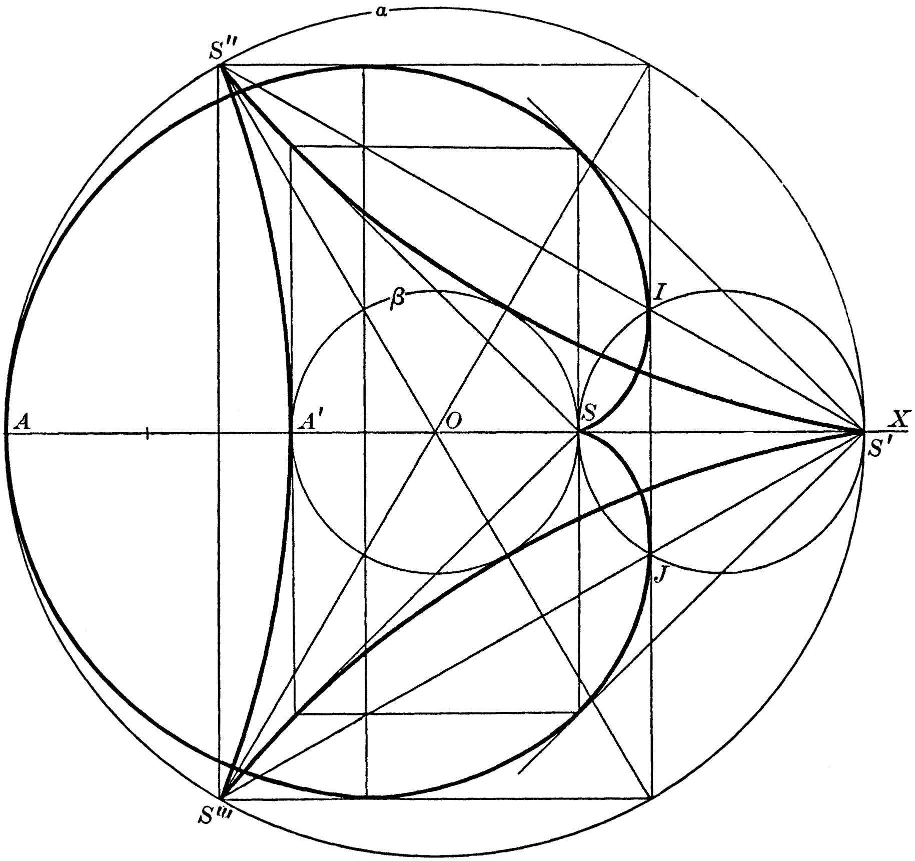
[*] J. Ozanam, *Dictionaire Mathématique ou Idée générale de Mathématique*, pp. 102–104 (Amsterdam, 1691).

[†] P. H. Fuss, *Correspondance math. et phys. du XVIII*$^{\text{ème}}$ *siècle*, vol. 1 (Euler and Goldbach), pp. 314–488 (published in 1843). See also *Acta eruditorum*, 1746.

[‡] Siebeck, *Crelle's Journal*, vol. 66 (1866), p. 359. — Frahm, *Schlömilch's Zeitschrift für Math.*, vol. 18 (1873), p. 363. — Wolstenholme, *Proc. London Math. Soc.*, vol. 4 (1873), p. 330;

Townsend and Delens; but it is hoped that the presentation of their results in a somewhat new and connected form, with the addition of several new theorems, may be suggestive and of interest.



Numerous properties of the cardioid and tricuspid will be assumed as known. For the former, reference may be made to my Dissertation; for the latter, to the bibliographies already cited.

Educational Times Reprint, vol. 55 (1891), p. 182; *Mathematical Problems*, 3rd edit. (1891), problem no. 1849, and note p. 251.—Townsend, *Educ. Times Reprint*, vol. 20 (1873), p. 34; vol. 22 (1875), p. 35.—Delens, *Journal de math. spéciales*, vol. 15 (1892), pp. 193–198.

* Archibald, *The Cardioide and some of its related curves*, Strassburg, 1900.

2. Consider a cardioid, C, and a tricuspid, T, placed in a very simple relation to each other, as follows:

Let $OA = 3a$ and $OA' = a$ be the radii of two concentric circles, α and β, respectively (see figure). Then:

If a circle with radius a roll on the outside of β, a point on its circumference will trace the *cardioid*, C, of the figure.*

The equations of C, referred to O as origin, are:

$$\begin{cases} x = a(2\cos\theta - \cos 2\theta), \\ y = a(2\sin\theta - \sin 2\theta). \end{cases}$$

Eliminating θ, we get

$$(x^2 + y^2)^2 - 6a^2(x^2 + y^2) + 8a^3x - 3a^4 = 0.$$

Introducing trilinear coordinates, this equation becomes:

If a circle with radius a roll on the inside of α, a point on its circumference will trace the *tricuspid*, T, of the figure.*

The equations of T, referred to O as origin, are:

$$\begin{cases} x = a(2\cos\theta + \cos 2\theta), \\ y = a(2\sin\theta - \sin 2\theta). \end{cases}$$

Eliminating θ, we get

$$(x^2 + y^2)^2 + 18a^2(x^2 + y^2) - 8ax(x^2 - 3y^2) - 27a^4 = 0.$$

Introducing trilinear coordinates, this equation becomes:

$$X^{-\frac{1}{2}} + Y^{-\frac{1}{2}} + Z^{-\frac{1}{2}} = 0, \qquad (A)$$

where

$$X = x + iy - a,$$
$$Y = x - iy - a,$$
$$Z = -a;$$

or,

$$\text{(I)} \quad \begin{aligned} x &= (X + Y - 2Z)/2, \\ y &= (X - Y)/2i, \\ a &= -Z. \end{aligned}$$

Equation (A) is the trilinear equation of C referred to an imaginary triangle, whose vertices are the three cusps of C; viz., the point $(a, 0)$ and the circular points at infinity.

where

$$X = x + \sqrt{3}y - 3a,$$
$$Y = x - \sqrt{3}y - 3a,$$
$$Z = -3a - 2x;$$

or,

$$\text{(II)} \quad \begin{aligned} x &= (X + Y - 2Z)/6, \\ y &= (X - Y)/2\sqrt{3}, \\ a &= -(X + Y + Z)/9. \end{aligned}$$

Equation (A) is the trilinear equation of T referred to a real triangle, whose vertices are the three cusps of T; viz., the real points $(3a, 0)$, $(-3a/2, 3\sqrt{3}a/2)$, and $(-3a/2, -3\sqrt{3}a/2)$.

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* The following relations between C and T may be worth noting: The axis (SA) of C is equal to the axis $(S'A')$ of T, and the length, $16a$, of the curve C is equal to the length of the curve T. The cuspidal chords of C are of constant length, $4a$, and the locus of their middle points is the base, β, of C. The portions of tangents to T intercepted by the curve are of constant length, $4a$, and the locus of their middle points is also the circle β.

3. By the aid of relations (I) and (II), the trilinear equation of any curve, referred to either of the triangles just described, is readily obtained from its equation in rectangular coordinates. For example:

1. From the equation $x = 3a/2$ of the double tangent of C, we get

1′. From the equation $a = 0$ of the line at infinity, we get

$$X + Y + Z = 0. \qquad (B)$$

2. For the circle

$$(x - 2a)^2 + y^2 = 0$$

through the cusp and the points of contact, I, J, of the double tangent, we get

2′. For the circle $\boldsymbol{a}$:

$$x^2 + y^2 = 9a^2,$$

through the three cusps, we get

$$XY + YZ + ZX = 0,$$

or,

$$(X + Y)^{\frac{1}{2}} + (Y + Z)^{\frac{1}{2}} + (Z + X)^{\frac{1}{2}} = 0. \qquad (C)$$

3. For the cuspidal tangents, $y = 0$, $x + iy = 0$, $x - iy = 0$, we get

3′. For the cuspidal tangents, $y = 0$, $y + \sqrt{3}x = 0$, $y - \sqrt{3}x = 0$, we get

$$X - Y = 0, \quad Y - Z = 0, \quad Z - X = 0.$$

These three lines intersect at the focus, O, of the cardioid.

These three lines intersect at the focus, O, of the tricuspid.

4. For the circle $\boldsymbol{a}$:

$$x^2 + y^2 = 9a^2,$$

we get

4′. For the ellipse

$$9(x + 2a)^2 + y^2 = 9a^2,$$

we get

$$XY - YZ - ZX = 8Z^2,$$

or,

$$(X - Z)^{\frac{1}{2}} + (Y - Z)^{\frac{1}{2}} + (X + Y - 8Z)^{\frac{1}{2}} = 0, \qquad (D)$$

or,

$$(X - Z)(Y - Z) = 9Z^2.$$

5. For the hyperbola
$$25(x+6a/5)^2-135y^2=81a^2,$$
we get

5′. For the circle β:
$$x^2+y^2=a^2,$$
we get

$$8(X+Y+Z)^2=27(XY+YZ+ZX),$$
or, $$(X+Y-8Z)^{\frac{1}{2}}+(Y+Z-8X)^{\frac{1}{2}}+(Z+X-8Y)^{\frac{1}{2}}=0, \qquad (E)$$
or, $$8(X^2+Y^2+Z^2)=11(XY+YZ+ZX).$$

6. For the circle β:
$$x^2+y^2=a^2,$$
we get

6′. For the ellipse
$$25(x-6a/5)^2-15y^2=81a^2,$$
we get

$$XY-YZ-ZX=0. \qquad (F)$$

7. For the circle
$$(3x-2a)^2+9y^2=a^2,$$
we get

7′. For the hyperbola
$$49(x-12a/7)^2-63y^2=81a^2,$$
we get

$$3XY-YZ-ZX=0. \qquad (G)$$

4. It has been shown by Salmon* that the trilinear equation of any tricuspidal quartic referred to a suitable triangle of reference is

$$X^{-\frac{1}{2}}+Y^{-\frac{1}{2}}+Z^{-\frac{1}{2}}=0.$$

Since either of two curves having the same trilinear equation but different triangles of reference may be obtained from the other by projection, it follows that from either the cardioid or the tricuspid we may, by a suitable projection, obtain the other ;† and further that from either we may obtain any tricuspidal quartic.‡

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* G. Salmon, *Higher Plane Curves*, 3rd edit. (1879), p. 258.

† The projective transformation which transforms C into T is given by the equations:

$$x=\frac{3ax'}{2x'+3a}, \qquad y=\frac{-\sqrt{3}iay'}{2x'+3a}.$$

‡ W. K. Clifford, in a paper : On Triangular Symmetry [*Educ. Times Reprint*, vol. 4 (1866), pp. 88-89; *Collected Math. Papers* (1882), pp. 412-414], attributes to Cayley a proof that "every three-cusped quartic can be projected into a hypocycloid of three cusps."

Notice that in any projection of the cardioid the circular points at infinity ($Y = Z = 0$, $X = Z = 0$) become cusps of the tricuspidal quartic into which the cardioid projects, while the line at infinity ($Z = 0$) becomes the line joining the cusps in question; on the other hand, in any projection of the tricuspid the line at infinity ($X + Y + Z = 0$) becomes the double tangent of the tricuspidal quartic into which the tricuspid projects, while the circular points at infinity become the points of contact I, J of the double tangent. Keeping these facts in mind, it is easy to obtain from any descriptive property of the cardioid [tricuspid]— or from any metric property which lends itself to projection—a property of any tricuspidal quartic, and incidentally of the tricuspid [cardioid].

5. Among many remarkable known properties of the cardioid and tricuspid we shall note a few which readily lend themselves to projection.

On the cardioid:

(a) Tangents to C at the ends of a cuspidal chord intersect orthogonally on the circle α.

(b) Normals to C at the ends of a cuspidal chord intersect orthogonally on the circle β.

(c) The locus of a point which so moves that the points of contact of tangents from it to C lie on a line, Δ, is the circle $(x - 2a)^2 + y^2 = a^2$,* passing through the cusp and the points of contact I, J of the double tangent.

(d) The locus of a point which so moves that the feet of the normals drawn from it to C lie in a line, Δ', is the circle $(3x - 2a)^2 + 9y^2 = a^2$.*

On the tricuspid:

(e) Tangents to T at the points where any tangent t cuts the curve, intersect orthogonally in a point P on the circle β; the third tangent which can be drawn from P to T meets t orthogonally on β.

(f) If three tangents are drawn to T from any point P on the circle β, the normals at their points of contact will intersect on the circle α at the point where α is cut by the line PO.

(g) Normals to T at the points where a tangent to T cuts the curve intersect orthogonally on the circle α.

(h) The locus of the point which so moves that the feet of normals drawn from it to T lie in a line, is the circle α.

6. By the aid of the relations established in §§ 2—4, the reader will

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*R Lachlan, *Educ. Times Reprint*, vol. 58 (1893), p. 42.

find it easy to extend these theorems of §5 to any tricuspidal quartic. For instance the following theorems are easily deduced:

(1°) The locus of a point which so moves that the points of contact of tangents from it to any tricuspidal quartic $X^{-\frac{1}{2}} + Y^{-\frac{1}{2}} + Z^{-\frac{1}{2}} = 0$ lie in a line Δ, is the conic (C') : $XY + YZ + ZX = 0$, through the cusps of the quartic and the points of contact I, J of its double tangent. The lines joining the triple focus of the quartic to I and J are tangent to this conic. For the special case of the tricuspid the locus in question is the circle a through the cusps.

(2°) Tangents to any tricuspidal quartic $X^{-\frac{1}{2}} + Y^{-\frac{1}{2}} + Z^{-\frac{1}{2}} = 0$ at the points P_2, P_3 when any tangent t cuts the curve, intersect in a point P on the conic (E') : $8(X + Y + Z)^2 = 27\,(XY + YZ + ZX)$, which touches the quartic in the points where the quartic is cut by its cuspidal tangents, and passes through I and J, the points of contact of the double tangent. These tangents PP_2 and PP_3 divide the line IJ harmonically. The third tangent from P to the quartic meets t in a point Q on (E'); the two tangents PQ and t also divide IJ harmonically. The line PO cuts the tangent t in its second point of intersection with (E'). Further, OI and OJ are tangent to (E'); hence the focus of the quartic is the pole of the double tangent with respect to either of the conics (C') or (E').

For the special case of the cardioid the conic (E') is a hyperbola.*

(3°) Quasi-normals† with respect to a pair of cusps B, C of a tricuspidal quartic, drawn at the ends of a chord through the third cusp, meet on the conic (F') : $XY - YZ - ZX = 0$ (which passes through the cusps), and divide BC harmonically.

(4°) If the feet of the three quasi-normals (with respect to I, J) which can be drawn from a point to a tricuspidal quartic lie in a line Δ', the locus of the point is a conic (G') : $3XY - YZ - ZX = 0$, passing through the cusps. For the special case of the cardioid the locus in question is a circle.

7. It is not difficult to prove that the envelope of the line Δ through the points of contact of the tangents drawn to the tricuspid from any point P

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*A result first given in 1866, by Siebeck, *loc. cit.*; but also in 1874 by Wolstenholme, *Educ. Times Reprint*, vol. 20, p. 34.

†A *quasi-normal* with respect to a pair of points is a line through the point of contact, P_1, of a tangent and forming with the tangent and the lines drawn from P_1 to the two points a harmonic pencil. A *quasi-evolute* is the envelope of the quasi-normals. Obviously a normal will always project into a quasi-normal with respect to the projections of the circular points at infinity, and an evolute into a quasi-evolute.

on the circle a (§ 5(c)) is the evolute of the tricuspid.[*] This curve may be obtained by enlarging T three times its linear dimensions and turning the resulting figure through 180°. The equation of this envelope is:

$$(X + 4Y + 4Z)^{-\frac{1}{2}} + (4X + Y + 4Z)^{-\frac{1}{2}} + (4X + 4Y + Z)^{-\frac{1}{2}} = 0. \quad (H)$$

Hence we have the theorem:

(5°) For any tricuspidal quartic, the envelope of the line Δ is another tricuspidal quartic (H'), tangent to the conic (C') at each of the cusps of the original quartic, and having the same cuspidal and double tangents as the original quartic, and the same points of contact I, J. Or, the envelope of Δ is the quasi-evolute (H') of the quartic, with respect to the points of contact I, J of the double tangent of the quartic.

We may finally deduce the following theorem from §5 (f), (g) with regard to the conic (C'):

(6°). The quasi-normals of a tricuspidal quartic (with respect to the points I, J) at three points P_1, P_2, P_3 intersect in a point Q on the conic (C') which passes through the cusps of the quartic; the quasi-normals QP_2, QP_3 divide IJ harmonically. Further, the points P, O, Q lie in a straight line which passes through one of the points where the tangent P_2P_3 cuts the conic (E').

8. *Problems.*

1. The points P_2, P_3 in §6 (2°) are said to be conjugate. Show that in the case of a cardioid these points lie on a circle through the real cusp S and the focus O of the curve; further that the tangents PP_2, PP_3 intersect the double tangent on this circle.[†] The equation of the circle being $x^2 + y^2 - ax - by = 0$ (b variable), deduce a general theorem for tricuspidal quartics.

2. The *orthoptic locus* of C is made up of the circle a (§5(a)) and the Limaçon of Pascal:

$$8(x^2 - y^2 - 9a^2)^2 + 54a^2(x^2 + y^2 - 9a^2) + 81(2x - 3a)a^3 = 0.\text{‡}$$

Hence, show that, if through a cusp S of any tricuspidal quartic $X^{-\frac{1}{2}} + Y^{-\frac{1}{2}} + Z^{-\frac{1}{2}} = 0$ any line be drawn, it meets the curve in two other points P_2, P_3, the tangents at which divide the line BC (joining the other cusps) harmonically; the locus of the intersection of these tangents is the conic (D'). The locus of the point of intersection of other tangents to the quartic which divide BC harmonically, is the bicuspidal quartic

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[*] Delens, *loc. cit.*

[†] *Cf.* Archibald, *Educ. Times Reprint*, problem no. 14586, vol. 75 (1901), p. 127.

[‡] Wolstenholme, *Proc. London Math. Soc.*, *loc. cit.*

$$8(X-Z)^2(Y-Z)^2-9Z^2(X-Z)(Y-Z)-81XYZ^2=0,$$

which is tangent to the tricuspidal quartic at the two points where the quasi-normals are also tangents.

3. With the notation of the last problem, show that the locus of the fourth harmonic point to P_2, S, P_3 is a cuspidal cubic with an inflexional tangent, defined by the equation $XY(X+Y)=Z(X-Y)^2$. Show that for the cardioid C, this cubic is the *Cissoid of Diocles*, with cusp at S and inflexional asymptote $x=3a$.*

4. The envelope of the sides of the triangles† formed by joining the points of contact of parallel tangents to a cardioid C, is the nodal cubic known as the *Trisectrix of Maclaurin*‡. Its equation may be written

$$r\sin 2\theta + a\sin 3\theta = 0, \quad \text{and} \quad r\cos(\theta/3) = a/2.$$

Hence, deduce a theorem for tricuspidal quartics.

9. Beside the cardioid and tricuspid, other definite examples of tricuspidal quartics have occurred in problems by Cayley§ and Wolstenholme.‖ To quote Problem 1698 of the latter may not be without interest in this connection :

"Through each point within a parabola $y^2=4ax$ it is obvious that at least one minimum chord can be drawn : prove that the part from which two minimum chords and one maximum can be drawn is divided from the part through which only one minimum can be drawn by the curve $(x-5a)^{-\frac{1}{2}} + (4x-6y+4a)^{-\frac{1}{2}} + (4x+6y+4a)^{-\frac{1}{2}} = 0$; etc."

Euler found (*loc. cit.*) a whole class of tricuspidal quartics in connection with a solution of the problem : "given a luminous point, find a curve such that rays of light twice reflected from it return to the starting point."

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* Archibald, *Educ. Times Reprint*, vol. 74 (1901), p. 53.

† These triangles have a constant area and a common centre of gravity, the triple focus of C.

‡ This remarkable curve has many beautiful properties and has been much studied, especially by the French. A bibliography and a proof of the above as well as other properties are given in my Dissertation, *loc. cit.* I have also given a bibliography in *L'Intermédiaire des Mathématiciens*, vol. 8 (1901), p. 10.

§ Cayley, *Educ. Times Reprint*, vol. 9 (1868).

‖ Wolstenholme, *Math. Problems*, *loc. cit.*, problems 1811, 1698, 1364, 1356.

Finally attention may be drawn to Problem 391 in an interesting collection by Ralph A. Roberts.*

10. It would be interesting, did space permit, to treat the nodal cubics (the reciprocals of tricuspidal quartics), whose equations may be thrown into the form†

$$X^{\frac{1}{3}} + Y^{\frac{1}{3}} + Z^{\frac{1}{3}} = 0,$$

by a method similar to that which we have used in the foregoing pages.

The results of the present paper were obtained four years ago. Within the past eighteen months a book by A. B. Basset‡ has appeared, making special note of the value of the method of projection in connection with the study of higher plane curves, whereas previously this theory had been largely confined to the treatment of conics.

SACKVILLE, N. B.
APRIL, 1903.

NOTE ON A PARTIAL DIFFERENTIAL EQUATION OF THE FIRST ORDER§

BY E. D. ROE, JR.

THIS note contains an elementary proof of the following familiar theorem :‖

A necessary and sufficient condition that a function $f(x_1, x_2, \ldots, x_n)$ *be a solution of the equation*

$$\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} + \cdots + \frac{\partial f}{\partial x_n} = 0 \tag{1}$$

is that f *be a function of the differences of the x's.*

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* *A Collection of Examples and Problems on Conics and some of the Higher Plane Curves*, Dublin, 1882.

† Salmon, *loc. cit*, p. 184.

‡ *An Elementary Treatise on Cubic and Quartic Curves*, Cambridge, 1901.

§ Presented to the American Mathematical Society at the Evanston meeting, September 2–3, 1902

‖ *Cf.* Faà di Bruno's *Binäre Formen*, p 29 and p. 46, where, however, only the sufficient condition, not the necessary condition, is proved. *Cf.* also G. Salmon, *Higher Algebra*, 4th edit., §§63–64; and Gordan and Noether, in *Math. Annalen*, Vol. 10 (1876), pp. 549–552.